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An explicit solution to the Skorokhod embedding problem for functionals of excursions of Markov processes

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Abstract

We develop an explicit non-randomized solution to the Skorokhod embedding problem in an abstract setup of signed functionals of excursions of Markov processes. Our setting allows us to solve the Skorokhod embedding problem, in particular, for the age process of excursions of a Markov process, for diffusions and their signed age processes, for Azéma's martingale and for Bessel processes of dimension smaller than 2.

This work is a continuation and an important generalization of Obłój and Yor [J. Obłój, M. Yor, An explicit Skorokhod embedding for the age of Brownian excursions and Azéma martingale, Stochastic Process. Appl. 110 (1) (2004) 83–110]. Our methodology is based on excursion theory and the solution to the Skorokhod embedding problem is described in terms of the Itô measure of the functional. We also derive an embedding for positive functionals and we correct a mistake in the formula of Obłój and Yor [J. Obłój, M. Yor, An explicit Skorokhod embedding for the age of Brownian excursions and Azéma martingale, Stochastic Process. Appl. 110 (1) (2004) 83–110] for measures with atoms.

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1. Introduction

The *Skorokhod embedding problem* was first introduced and solved by Skorokhod [51], where it served to realize a random walk as a Brownian motion stopped at a sequence of stopping times. The original problem can be formulated as follows: given a Brownian motion (B_t) and a centered probability measure μ with finite variance, find an integrable stopping time T which embeds μ in $B : B_T \sim \mu$. Skorokhod's original solution used randomized stopping times, however subsequent solutions (e.g. [49,1,37,5]) mostly required T to be a stopping time relative to the natural filtration of the process. They also assumed only that μ is centered and the condition of integrability of T was replaced with the requirement that $(B_{t\wedge T})$ is a uniformly integrable martingale or equivalently that T is minimal. We recall that T is minimal for process (X_t) if for a stopping time S with $S \leq T, X_S$ has the same law as X_T implies $S = T$ a.s. (cf. [29,13,16], [32, Sec. 8]). We refer to our survey paper [32] for background on Skorokhod embeddings.

Quite amazingly, the Skorokhod embedding problem has continued to stimulate probabilists for over 40 years now and has actually seen a recent revival (e.g. [12,15,14,32,34,39]). It also found new applications in the field of mathematical finance, such as pricing and hedging of lookback and barrier options [19,8]. A new explicit solution, in a discontinuous setup, was proposed by Obłój and Yor [35]. The authors described an explicit and non-randomized solution to the Skorokhod embedding problem for the age of Brownian excursions, or more generally for positive functionals of Brownian excursions. However, they were only able to develop a randomized solution for the Azéma martingale. Their work left two open challenges: firstly to extend the methodology to an abstract Markovian setting, and secondly to extend the methodology to provide an explicit, non-randomized embedding for Azéma's martingale, or more generally for signed functionals of excursions. The latter is very natural, as already argued in [35], and it actually motivated the present study. Indeed, Azéma's martingale – the projection of Brownian motion on the filtration generated by the signs – is an important process which, even though quite simple, inherits a number of important properties from Brownian motion and finds various applications (e.g. [11]). So far no explicit non-randomized solution to the Skorokhod embedding problem for this process existed.

In this paper we solve both of the aforementioned open problems resulting from Obłój and Yor [35]. We present an explicit non-randomized solution to the Skorokhod embedding problem for signed functionals of excursions of a Markov process. This abstract solution contains embeddings for Azéma's martingale, or signed age processes in general, for certain Bessel processes and for real-valued diffusions, to mention some examples. The stopping times we study here can be thought of as *two-sided* generalizations of the stopping times introduced by Obłój and Yor [35] or, going back to the origins, of the stopping times introduced by Azéma and Yor [1]. We recall that Azéma and Yor [1] studied stopping times of the form $T = \inf\{t \geq 0 : B_t \geq \varphi(\sup_{s\leq t} B_s)\}$, where φ is an increasing function. In [35] Obłój and Yor considered their analogue with the maximum process replaced with the local time at zero: $T = \inf\{t : F_t \geq \varphi(L_t)\}$, where F_t was a positive functional of excursions (see Section 2.2 below). The advantage of such stopping times is that $F_T = \varphi(L_T)$ and thus to describe the law of F_T it suffices to describe the law of L_T. Here we propose to investigate *two-sided* stopping times, namely the stopping times of the form $T = \inf\{t : F_t \leq -\varphi_-(L_t) \text{ or } F_t \geq \varphi_+(L_t)\}$, where (F_t) is a process generated by a certain (signed) functional over excursions of an underlying Markov process. Similar stopping times, in a Brownian setting, were considered by Jeulin and Yor [22] and Vallois [53], and also recently, in more generality, by Cox and Hobson [14].

Given a probability measure μ, we describe two increasing functions $\varphi_{+/-}$ such that F_T has the distribution μ.

This paper is organized as follows. We first introduce the necessary notation and objects, in particular we discuss, in Section 2.1, the excursion process of a Markov process and in Section 2.2 we define the class of functionals we will consider and we clarify the terminology used throughout the paper. Then in Section 3 we present our main results, in Theorem 1 for signed functionals, and in Theorem 4 for positive functionals. The latter corrects a mistake found in the formula of Obłój and Yor [35]. In the subsequent two sections we develop applications of these results. Section 4 presents applications of Theorem 1 and contains in particular an explicit solution to the Skorokhod embedding problem for the Azéma martingale, for the Cauchy principal value associated with Brownian local times (over one excursion), for skew Brownian motion and for Brownian motion itself. Section 5 contains applications of Theorem 4 and develops explicit solutions to the Skorokhod embedding problem for Bessel processes of dimension $\delta \in (0, 2)$ and their age processes, as well as for the age process of excursions of Cox–Ingersoll–Ross processes. In Section 6 we comment on embeddings for measures with atoms, which are not covered by Theorem 1. Finally, in Section 7 we prove Theorems 1 and 4.

2. Main objects and the setup

We start by introducing the basic objects and notation that will be ubiquitous in this paper. We place ourselves in a general Markovian context and we follow closely Bertoin [4] to which we refer for all the details. Specific notation connected with examples or particular cases will be introduced later, when necessary.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider $X = (X_t : t \geq 0)$ a stochastic process taking values in some Polish space (E, ρ) and having right-continuous sample paths. We assume X is a 'nice' Markov process in the sense of Bertoin [4]. Furthermore we suppose 0 is regular and instantaneous for X, and that it is recurrent. We write $\mathbb{P}$ for $\mathbb{P}_0$ (where $\mathbb{P}_0(X_0 = 0) = 1$) and $\mathbb{E}$ for $\mathbb{E}_0$, the expectation under $\mathbb{P}_0$. The process $B = (B_t : t \geq 0)$ denotes always a standard Brownian motion.

For a probability measure μ on $\mathbb{R}$, we denote its left-continuous tail by $\overline{\mu}(t) = \mu([t, \infty))$ and its support by $\mathrm{supp}(\mu)$. The lower and upper bounds of the support are denoted respectively a_μ and b_μ. Dirac's delta measure at point y is denoted δ_y.

All functions considered in the sequel are assumed to be Borel measurable.

2.1. Markovian local time and excursions

We introduce now the local time and the excursion process of X which will be our main tools in this paper. We follow Bertoin [4] (see [7] for an alternative approach based on potential theory). The set of zeros of X is denoted $\mathcal{Z} = \{t : X_t = 0\}$. The last zero before time t and the first zero after time t are denoted respectively g_t and d_t that is $g_t, d_t \in \overline{\mathcal{Z}}$ with $g_t \leq t < d_t$ and $X_s \neq 0$ for $u \in (g_t, d_t)$. An interval of the form (g_t, d_t) is called an excursion interval. The local time at 0 of X is denoted $L = (L_t : t \geq 0)$ and we recall that it is adapted to the filtration generated by zeros of X (cf. [4, Thm. IV.4]). The right-continuous inverse of the local time $\tau = (\tau_l : l \geq 0)$, $\tau_l = \inf\{t \geq 0 : L_t > l\}$, is a subordinator with infinite Lévy measure Λ^X (in particular $L_\infty = \infty$ a.s.). Note that the difference $(\tau_l - \tau_{l-})$ is just the length of the constancy period of L at the level l, which in turn is the lifetime of the corresponding excursion. Thus $\overline{\mathcal{Z}}$ is the closure of the range of $(\tau_l : l \geq 0)$.

The space of excursions is defined as $U = \{\epsilon : \mathbb{R}_+ \to E : \exists\ V(\epsilon), \epsilon(r) \neq 0$, for $r \in (0, V(\epsilon))$ and $\epsilon(r) = 0$ for $r \geq V(\epsilon)\}$. The excursion process of X, $e = (e_l : l \geq 0)$ takes values in the space U of excursions with an additional isolated point Υ, that is $U \cup \{\Upsilon\}$, and is given by

$$e_l = (X_{\tau_{l-}+r}, 0 \leq r \leq \tau_l - \tau_{l-}) \quad \text{if } \tau_{l-} < \tau_l, \tag{1}$$

and $e_l = \Upsilon$ otherwise. One of the most important results for us, going back to the fundamental paper of Itô [20], is that the above process is a Poisson point process with a certain characteristic measure n. This measure, called the Itô measure, is uniquely determined up to a multiplicative constant factor. We will show however that our results are invariant under multiplication of the excursion measure by a constant.

The Lévy measure of the subordinator τ can be easily deduced from Itô's measure n. More precisely, as the lifetime of an excursion corresponds to the height of the jumps of τ, we have $n(V(\epsilon) > a) = \Lambda^X((a, \infty))$, $a > 0$ (cf. [4, p. 117]). Similar measures, with V replaced by a general functional F, will be of prime importance in the sequel.

2.2. *Signed functionals of excursions*

We introduce now the main objects of this work, namely the signed functionals of Markovian excursions for which we want to solve the Skorokhod embedding problem. The functionals we are interested in are real-valued, continuous and monotone (note that an excursion is not assumed to be continuous). In particular, for a fixed excursion, they either stay positive or negative.

More precisely let $F : U \times \mathbb{R}_+ \to \mathbb{R}$. We will write both $F(\epsilon, r)$ and $F(\epsilon)(r)$, the latter being used to stress the time-dependence, with a particular excursion ϵ being fixed. For a fixed excursion $\epsilon \in U$, $F(\epsilon)$ is a *monotone, continuous* function. It starts at zero, $F(\epsilon)(0) = 0$, and is constant after the lifetime of ϵ, that is $F(\epsilon)(r) = F(\epsilon)(V(\epsilon))$ for any $r \geq V(\epsilon)$.

Since we want the process induced by the functional to be adapted, we impose the condition that the value $F(\epsilon, r)$ is determined from the values of the excursion up to time r : $F(\epsilon, r) = F((\epsilon_s : s \leq r), r)$, that is for any $r \geq 0$ and $\epsilon, \epsilon' \in U$ such that $(\epsilon_s : s \leq r) = (\epsilon'_s : s \leq r)$, $F(\epsilon, r) = F(\epsilon', r)$. We set F of the trivial excursion Υ to be zero: $F(\Upsilon) \equiv 0$, and assume that $|F(\epsilon, V(\epsilon))| > 0$ for all $\epsilon \in U$.

A functional F induces a process in time $(F_t : t \geq 0)$, the value F_t given as the functional F of the excursion straddling time t evaluated at the age of this excursion, that is

$$F_t = F\left(e_{L_t}\right)(t - g_t). \tag{2}$$

Note that the set of zeros of the process $(F_t : t \geq 0)$ is equal to $\mathcal{Z}$, the set of zeros of X, and thus the local time (L_t) is adapted to the natural filtration of (F_t). The process of terminal values $(F(e_l, V(e_l)) : l \geq 0)$ is a Poisson point process and we denote its characteristic measure by n_F. This measure is just the image of the excursion measure n of X, by $\epsilon \to F(\epsilon, V(\epsilon))$.

Functional F is called positive if $F : U \times \mathbb{R}_+ \to \mathbb{R}_+$, that is if $F_t \geq 0$ for $t \geq 0$, or equivalently if $n_F((-\infty, 0)) = 0$.

In this paper, when we speak of the process F we always mean the process $(F_t : t \geq 0)$, which is adapted to the natural filtration of X. The natural filtration of F designates the natural filtration of the process $(F_t : t \geq 0)$ that is $(\sigma\{F_s : s \leq t\} : t \geq 0)$. When speaking of the characteristic measure of the functional associated with (F_t) we mean the measure n_F.

We present now some examples of the functionals described above. Suppose that X is a one-dimensional diffusion on (l, u), $l < 0 < u$. In particular X has continuous sample paths, excursions are either positive or negative and thus we can speak of $\mathrm{sgn}(\epsilon)$, the sign of an excursion ϵ. The first examples we present are connected with the age of excursion:

$$A(\epsilon)(r) = \mathrm{sgn}(\epsilon)\,(r \wedge V(\epsilon)) \quad \text{and}$$
$$\alpha(\epsilon)(r) = \frac{\mathbf{1}_{\mathrm{sgn}(\epsilon)=1}}{n_A((A(\epsilon,r),\infty))} - \frac{\mathbf{1}_{\mathrm{sgn}(\epsilon)=-1}}{n_A((-\infty, A(\epsilon,r)))}. \tag{3}$$

They yield the signed age process $A_t = \mathrm{sgn}(X_t)(t - g_t)$ and the process

$$\alpha_t = \frac{\mathbf{1}_{X_t>0}}{n_A((t-g_t,\infty))} - \frac{\mathbf{1}_{X_t<0}}{n_A((-\infty, g_t - t))} \tag{4}$$

which is a martingale in the filtration generated by zeros of X, which is just the natural filtration of (α_t). Thus, the functional α yields a natural family of martingales (α_t) which are associated with the age process (A_t). That (α_t) is a martingale is well known for diffusions on a natural scale (see [45,46], [43, Rk. 3]) but is generalized to our setup upon taking the scale function s such that $s(0) = 0$. Then X and $s(X)$, which is a diffusion on the natural scale, have the same zeros and thus the same local times at zero and characteristic measures n_A (up to a multiplicative constant). The characteristic measure of α is easily seen to satisfy $n_\alpha(\mathrm{d}v) = \frac{\mathrm{d}v}{v^2}$. Note also that the measure n_A satisfies $n_A((-\infty,-x) \cup (x,\infty)) = n_V((x,\infty)) = \Lambda^X((x,\infty))$, $x > 0$. Furthermore, the measure n_A is absolutely continuous with respect to the Lebesgue measure and its support is given as $\mathbb{R}$, $\mathbb{R}_+$ or $\mathbb{R}_-$ (cf. [21, Sec. 6.2]). In particular we can replace the open intervals in (4) with closed ones. Profound studies of the Itô measure n, in particular of the Lévy measure Λ^X, were made via Krein's string theory. For more details on the measure Λ^X we refer the reader to [24, pp. 71,77] and [26,2] (see [18] for a recent account and further references).

When $X = B$ is a Brownian motion, then $\alpha_t = \mathrm{sgn}(B_t)\sqrt{2\pi(t-g_t)}$, in which we recognize (up to a constant multiplicative factor) the celebrated Azéma martingale, which is the projection of B on the filtration generated by its signs. Similar projection properties hold in the general setting (see Section 3.3). Embeddings for these processes are discussed in detail in Section 4.

We can generalize upon (3) in the abstract setting. Notice that even when we can not speak of the sign of an excursion $\epsilon \in U$, the sign of the function $F(\epsilon)$ is well defined. Thus, for a given functional F we can define its scaled version G^F through

$$G^F(\epsilon, r) = \frac{\mathbf{1}_{\mathrm{sgn}(F(\epsilon))=1}}{n_F((F(\epsilon,r),\infty))} - \frac{\mathbf{1}_{\mathrm{sgn}(F(\epsilon))=-1}}{n_F((-\infty, F(\epsilon,r)))}. \tag{5}$$

When the measure n_F is absolutely continuous with respect to the Lebesgue measure, it is immediate that $n_G(\mathrm{d}x) = \mathrm{d}x/x^2$, $x \neq 0$.

A large family of functionals is given by

$$F^\gamma_\beta(\epsilon)(r) = \mathrm{sgn}(\epsilon)\left(\int_0^{r\wedge V(\epsilon)} |\epsilon(s)|^\beta \mathrm{d}s\right)^\gamma, \tag{6}$$

where γ, β are taken such that F_t can be well defined. The functional F^1_{-1} is related with the Cauchy principal value associated with Brownian local times and will be studied in Section 4. The family (6) contains also the functional F^1_1 which is connected with the area processes $\int_0^t |B_s|\mathrm{d}s$ and $\int_0^t B_s^+\mathrm{d}s$, objects of great interest ever since the works of Cameron and Martin [9] and Kac [23] (see [38] for a study using excursion theory and some applications in statistics).

The signed extrema process can be obtained taking

$$M(\epsilon)(r) = \mathrm{sgn}(\epsilon) \sup_{s \le r \wedge V(\epsilon)} |\epsilon(s)|, \tag{7}$$

which yields $M_t = \mathrm{sgn}(X_t) \sup_{g_t \le s \le t} |X_s|$. This is a particularly important functional for us as our stopping times T, defined below in (9), satisfy $X_T = M_T = \mathbf{1}_{X_T \ge 0} \sup_{s \le T} X_s + \mathbf{1}_{X_T < 0} \inf_{s \le T} X_s$. Thus if we describe the distribution of M_T we automatically describe the distribution of X_T.

When the underlying Markov process X has the self-similarity property, $(X_{ct} : t \ge 0) \stackrel{\mathcal{L}}{=} (c^\kappa X_t : t \ge 0)$ for some $\kappa \in \mathbb{R}$, most of the functionals exemplified above fall into an important class of *homogeneous functionals*. More precisely, following [10,44], we say that F is a θ*-homogeneous functional* of X if

$$F(\epsilon, V(\epsilon)) = V(\epsilon)^\theta F\left(V(\epsilon)^{-\theta}\tilde{\epsilon}, 1\right), \tag{8}$$

where $\epsilon = (\epsilon_r : r \ge 0) \in U$ and $\tilde{\epsilon} = (\epsilon_{rV(\epsilon)} : r \ge 0)$, so that $V(\tilde{\epsilon}) = 1$. The age and signed extrema are κ-homogeneous functionals and F^γ_β is $\gamma(\beta\kappa + 1)$-homogeneous. The characteristic measures of homogeneous functionals are easier to calculate thanks to the scaling property of F inherited from X. This was exemplified in [35].

For Brownian motion, $X = B$, we can obtain in our setup the process $p_t = \int_{g_t}^t \mathrm{d}s/B_s$ but not the process $P_t = \int_0^t \mathrm{d}s/B_s$ (understood properly, cf. Section 4). However, we have to bear in mind that changing X might allow us to treat such processes. In particular we could consider $X_t = (R^{(q)}_t, K_t)$ where $R^{(q)}$ is a Bessel process with index $q \in (1, 2)$ and K appears in the classical Dirichlet process decomposition $R^{(q)}_t = B_t + \frac{(1-q)}{2} K_t$, and is locally of zero energy. An independent definition of K_t goes through the family of local times of $R^{(q)}$ and we can also write $K_t = \int_0^t \mathrm{d}s/R^{(q)}_s$ where the integral can be taken as Cauchy principal value or *partie finie* in Hadamard's sense associated with local times of X (see [56, Sec. 10.4] and [3] for details). Bertoin [3] showed that $(0, 0)$ is regular for X and described Itô's measure of excursions of X. As explained above, we could use the signed extrema functional (of the first coordinate of excursions of X away from $(0, 0)$) to control the process (K_t). We note also that the process K_t is actually a time-changed version of $\int_0^t \mathrm{d}s/|B_s|^{1+1/q}$. We will not go further into this domain as it is not our aim here, but we hope these examples have served to illustrate the generality of our setup.

We assumed above $|F(\epsilon, V(\epsilon))| > 0$. If one agrees to work with stopping times relative to a larger filtration, as the natural filtration of X, then our study generalizes instantly to functionals F which can be equivalent to zero at some excursions. An important example of such a functional is given by the sojourn time of a one-dimensional diffusion X, during an excursion, above a certain level λ : $F^\lambda(\epsilon, r) = \int_0^{r\wedge V(\epsilon)} \mathbf{1}_{\epsilon_s \ge \lambda} \mathrm{d}s$. The process $F^\lambda_t = \int_{g_t}^t \mathbf{1}_{X_s \ge \lambda} \mathrm{d}s$ is closely linked with the sojourn time for X, $\int_0^t \mathbf{1}_{X_s \ge \lambda} \mathrm{d}s$. The latter is an additive functional studied by a number of authors (cf. [27,28,52]) and the characteristic measure n_{F^λ} can be calculated.

We indicated above how the embedding for (M_t) yields also an embedding for (X_t). In fact, using our approach one can even weaken the continuity assumption on M and still develop an embedding for X. In this manner we could obtain an explicit solution to the embedding problem for spectrally one-sided Lévy processes and their reflected versions (see also [39]). Naturally, new issues arise linked for example with the minimality of stopping times. We plan to pursue these topics in a separate work.

3. Main results

The problem we want to solve here is the following: for a given functional F, as in Section 2.2, and a probability measure μ on $\mathbb{R}$, describe explicitly a minimal stopping time T, in the natural filtration of F, such that $F_T \sim \mu$. The construction can require certain properties of the measure μ and we will say that μ is admissible if it has these properties. Note that F needs not to be Markovian and thus we cannot rely on Rost's [50] criteria to determine the class of measures which can be embedded in F.

For $\varphi_-, \varphi_+ : \mathbb{R}_+ \to \mathbb{R}_+$ two non-decreasing functions define

$$T^F_{\varphi_-,\varphi_+} = \inf\{t > 0 : F_t \notin (-\varphi_-(L_t), \varphi_+(L_t))\} \tag{9}$$

which is a stopping time in the natural filtration of F. Note that when $\varphi_{+/-}$ are strictly increasing, their inverse functions $\psi_{+/-}$ are continuous and we have $T^F_{\varphi_-,\varphi_+} = \inf\{t > 0 : \psi_-(-F_t) \geq L_t$ or $\psi_+(F_t) \geq L_t\}$.

For a given probability measure μ on $\mathbb{R}$ we will search to specify the functions φ_- and φ_+ such that $F_{T^F_{\varphi_-,\varphi_+}} \sim \mu$. We will write, when we want to stress a particular dependence, $T^F_{\varphi_-,\varphi_+} = T^F = T^F_\mu$. We will also drop the superscript F when no confusion about the functional under consideration is possible.

3.1. Signed functionals

With a non-atomic probability measure μ on $\mathbb{R}$ and a functional F, as described in Section 2.2, we associate the following functions:

$$D_\mu(y) = \int_{[0,y]} \frac{\mathrm{d}\mu(s)}{n_F([s,+\infty))} \quad \text{and} \quad G_\mu(x) = \int_{[x,0]} \frac{\mathrm{d}\mu(s)}{n_F((-\infty,s])}, \tag{10}$$

for $y \geq 0$ and $x \leq 0$, and where n_F is the characteristic measure of the Poisson point process of the terminal values of F, as defined in Section 2.2. The inverses D_μ^{-1}, G_μ^{-1} are taken right-continuous, $D_\mu^{-1}|_{[D_\mu(b_\mu),\infty)} = \infty$, $G_\mu^{-1}|_{[G_\mu(a_\mu),\infty)} = -\infty$. We make the following fundamental assumptions

$$x \in \operatorname{supp}(\mu) \Rightarrow n_F((-\infty,x]) \cdot n_F([x,\infty)) > 0, \tag{11}$$

$$D_\mu(\infty) = \int_0^\infty \frac{\mathrm{d}\mu(s)}{n_F([s,+\infty))} = \int_{-\infty}^0 \frac{\mathrm{d}\mu(s)}{n_F((-\infty,s])} = G_\mu(-\infty), \tag{12}$$

which ensures that the inverse functions $D_\mu^{-1}(G_\mu(\cdot))$ and $G_\mu^{-1}(D_\mu(\cdot))$ are well defined. Thus, for $y, z \geq 0$, we can define

$$\psi_+(y) = \int_0^y \frac{\mathrm{d}\mu(s)}{n_F([s,+\infty))(1+\overline{\mu}(s)-\overline{\mu}(G_\mu^{-1}(D_\mu(s))))} \tag{13}$$

$$\psi_-(z) = \int_{-z}^0 \frac{\mathrm{d}\mu(s)}{n_F((-\infty,s])(1+\overline{\mu}(D_\mu^{-1}(G_\mu(s)))-\overline{\mu}(s))} \tag{14}$$

and $\psi_{-|\mathbb{R}_-} \equiv \psi_{+|\mathbb{R}_-} \equiv 0$. As a consequence of (37) below, we will see that $\psi_+(y) = \psi_-(z) = \infty$ for $y \geq b_\mu$ and $z \geq -a_\mu$. Define the left-continuous inverses $\varphi_-, \varphi_+ : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\varphi_{+/-}(y) := \psi_{+/-}^{-1}(y) = \inf\{x \geq 0 : \psi_{+/-}(x) \geq y\} \tag{15}$$

$\varphi_-(0) = \varphi_+(0) = 0$. When there are two or more functionals to be considered, we will often add a superscript F to the functions defined in (10)–(15) to avoid any confusion. We are ready to present our main result.

Theorem 1. *Let F be a functional as defined in Section 2.2 and μ a non-atomic probability measure on $\mathbb{R}$ such that $\mu(\mathbb{R}_-) > 0$, $\mu(\mathbb{R}_+) > 0$, and (11) and (12) hold. Then $T^F_{\varphi_-,\varphi_+}$ defined in (9), where $\psi_{+/-}, \varphi_{+/-}$ are given by (13)–(15), is an a.s. finite stopping time and it solves the Skorokhod embedding problem for $(F_t : t \geq 0)$, i.e. $F_{T^F_{\varphi_-,\varphi_+}} \sim \mu$. Furthermore, $T^F = T^F_{\varphi_-,\varphi_+}$ is minimal and $F_{T^F} = \sup_{t\leq T^F} F_t \cdot \mathbf{1}_{F_{T^F}\geq 0} + \inf_{t\leq T^F} F_t \cdot \mathbf{1}_{F_{T^F}<0}$.*

In particular, when μ and F are symmetric (i.e. invariant under $x \to -x$) we have $T^F = \inf\{t > 0 : \psi(|F_t|) \geq L_t\}$, where $\psi(y) = \int_0^y \frac{\mathrm{d}\mu(s)}{2\overline{\mu}(s)n_F([s,\infty))}$.

Remarks. The solution described in Theorem 1 depends on F only through its characteristic measure n_F. We note that in the particularly important case when $n_F(\mathrm{d}x) = \mathrm{d}x/x^2$ the formulae (10)–(14) simplify considerably.

The solution in the special case of Brownian motion and the signed extrema functional (7) yields a solution to the Skorokhod embedding problem for Brownian motion which coincides with the solution of Vallois [53]. See Section 3.3 below for details.

The functions $\psi_{+/-}$ are taken increasing. Using the same methodology, we could also develop an analogue embedding but with $\psi_{+/-}$ decreasing. In a Brownian motion setup this was done by Vallois [54] and we will come back to this issue after Proposition 5.

We specify now to the functionals related to the age process of excursions. In this case we dispose of a family of martingales (4) which allows us to understand better the condition (12).

Proposition 2. *Let $(X_t : t \geq 0)$ be a one-dimensional diffusion on $(l, u), l < 0 < u, X_0 = 0$ a.s., and $A_t = \mathrm{sgn}(X_t)(t - g_t)$ be the signed age process of excursions of X. Recall that its scaled version $(\alpha_t : t \geq 0)$, given in (4), is a martingale. For a non-atomic probability measure μ on $\mathbb{R}$ there exists a stopping time S in the natural filtration of (A_t) such that $A_S \sim \mu$ and $(\alpha_{t\wedge S} : t \geq 0)$ is a uniformly integrable martingale if and only if $D^A_\mu(\infty) = G^A_\mu(-\infty) < \infty$.*

If $D^A_\mu(\infty) = G^A_\mu(-\infty) < \infty$ then $T^A_{\varphi_-,\varphi_+}$, given in (9), is an a.s. finite stopping time in the natural filtration of (A_t), $A_{T^A_{\varphi_-,\varphi_+}} \sim \mu$, and $(\alpha_{t\wedge T^A_{\varphi_-,\varphi_+}} : t \geq 0)$ is a uniformly integrable martingale.

The proof of Proposition 2 is presented in Section 4. We just note here that the uniform integrability condition is reminiscent of the standard Skorokhod embedding problem and that $\mathbb{E}\alpha_{T^A_{\varphi_-,\varphi_+}} = 0$ is equivalent to $D^A_\mu(\infty) = G^A_\mu(-\infty) < \infty$, which is a version of (12).

Typically, one obtains a solution to the Skorokhod embedding problem for diffusions, by adapting a solution developed for Brownian motion (cf. [32, Sec. 8]). Here we obtain an embedding for diffusions directly from Theorem 1.

Proposition 3. *Let $(X_t : t \geq 0)$ be a one-dimensional diffusion on $(l, u), l < 0 < u, X_0 = 0$ a.s. Assume that X is on natural scale and choose the classical normalization of the local time L given by $L_t = |X_t| - \int_0^t \mathrm{sgn}(X_s)\mathrm{d}X_s$ a.s. Let M be the signed extrema functional given by (7), μ a non-atomic centered probability measure on (l, u), and $\psi_{+/-}, \varphi_{+/-}$ defined via (13)–(15). Then the characteristic measure of M is given by $n_M(\mathrm{d}x) = \frac{\mathrm{d}x}{2x^2}, x \neq 0$. The stopping time $T^M_{\varphi_-,\varphi_+}$, defined in (9), satisfies $M_{T^M_{\varphi_-,\varphi_+}} = X_{T^M_{\varphi_-,\varphi_+}} \sim \mu$, and $(X_{t\wedge T^M_{\varphi_-,\varphi_+}} : t \geq 0)$ is a uniformly integrable martingale.*

The condition that μ is centered, that is $\int_{\mathbb{R}} |x| \mathrm{d}\mu(x) < \infty$ and $\int_{\mathbb{R}} x \mathrm{d}\mu(x) = 0$, is a necessary condition for existence of a stopping time S such that $X_S \sim \mu$ and $(X_{t\wedge S} : t \geq 0)$ is a uniformly integrable martingale (cf. [32, Sec. 8]). It is equivalent to $D^M_\mu(\infty) = G^M_\mu(-\infty) < \infty$. The proof of Proposition 3 is given in Section 4.

3.2. Positive functionals

In [35], which inspired the present study, the authors considered positive functionals of Brownian motion with some particular interest placed upon the functionals F with $n_F(\mathrm{d}x) = \frac{\mathrm{d}x}{x^2}\mathbf{1}_{x>0}$. We now extend this to deduce explicit formulae for arbitrary measures. In Section 5 we will develop applications for functionals F with $n_F(\mathrm{d}x) = \frac{\mathrm{d}x}{x^2}\mathbf{1}_{x>0}$ and provide in (25) a corrected form of the *dual Hardy–Littlewood* function introduced in [35].

Theorem 4. *Let F be a positive functional, as defined in Section 2.2, and μ a probability measure on $\mathbb{R}_+$ with $\mu(\{0\}) = 0$ and $n_F([y, \infty)) > 0$ for $y \in \operatorname{supp}(\mu)$. Define*

$$\psi_\mu(y) = -\int_{[0,y)} \frac{\mathrm{d}\,(\ln \overline{\mu}(s))}{n_F([s,\infty))} \tag{16}$$

and φ_μ its right-continuous inverse. Then the stopping time

$$T^F_{\varphi_\mu} = \inf\{t > 0 : F_t \geq \varphi_\mu(L_t)\} \tag{17}$$

is a.s. finite and solves the Skorokhod embedding problem for F, i.e. $F_{T^F_{\varphi_\mu}} \sim \mu$. Furthermore, $T^F_{\varphi_\mu}$ is minimal and $\sup_{t\leq T^F_{\varphi_\mu}} F_t = F_{T^F_{\varphi_\mu}}$.

For μ a probability measure on $\mathbb{R}_+$ with $\mu(\{0\}) = \varsigma > 0$ define $\tilde{\mu} = \mu - \varsigma(\delta_0 - \delta_\infty)$. Then $\psi_{\tilde{\mu}}(\infty) < \infty$ and the stopping time

$$\tilde{T}^F_\mu = T^F_{\varphi_{\tilde{\mu}}} \wedge \inf\{t > 0 : L_t = \psi_{\tilde{\mu}}(\infty)\} \tag{18}$$

embeds μ in F, i.e. $F_{\tilde{T}^F_\mu} \sim \mu$.

We prove below that the law of $L_{T^F_{\varphi_\mu}}$ is absolutely continuous with respect to the Lebesgue measure so taking φ_μ left-continuous does not affect our solution.

The second part of the theorem provides a way of dealing with an atom at zero of μ which follows the idea of Vallois [53]. Note that if we applied the first part of the theorem for μ with $\mu(\{0\}) > 0$, we would have $T^F_{\varphi_\mu} = 0$ a.s. Another way of dealing with an atom at zero is to use a standard external randomization (cf. [32, Sec. 6.1]).

Finally we note, that upon taking $G_\mu \equiv 0$, for probability measure μ with $\mu(\mathbb{R}_-) = 0$, Theorem 4 for non-atomic measures can be seen as a direct corollary of Theorem 1. In particular, the expression (16) for ψ_μ then coincides with the expression (13) for ψ_+.

3.3. Links with martingale theory

We come back to the remarks below Theorem 1 and establish a link with the solution to the Skorokhod embedding problem developed by Vallois [53]. More generally, as we rely on excursion theory throughout the paper, we mention possible martingale theory arguments and establish a link with the study of Jeulin and Yor [22] which was the cornerstone of the work of Vallois [53]. We recall that Obłój and Yor [35] presented martingale theory arguments and already established there the link with Vallois' solution for symmetric measures. Here we complete this discussion.

In Proposition 3 we showed how to apply Theorem 1 to obtain an embedding for a one-dimensional diffusion. Consider then the particular case of Brownian motion. The stopping time in (9) can be written as $T_{\varphi_-,\varphi_+} = \inf\{t > 0 : B_t = -\varphi_-(L_t) \text{ or } B_t = \varphi_+(L_t)\}$. We recognize instantly the form of the stopping times considered by Jeulin and Yor [22] and Vallois [53]. Recall that as we work with the signed extrema functional which obeys $n_M(\mathrm{d}x) = \mathrm{d}x/2x^2$ we have $D_\mu(y) = 2\int_0^y s\mathrm{d}\mu(s)$ and $G_\mu(x) = -2\int_x^0 s\mathrm{d}\mu(s)$. These coincide with the functions $2\rho^+$ and $2\rho^-$ defined in [53] and indeed one can check that our solution coincides with the solution of Vallois.

Consider (X_t) a one-dimensional diffusion, martingale on some open interval containing zero and let $(\mathcal{H}_t)$ be the right-continuous version of $(\mathcal{F}_{g_t})$. Rainer [45] showed that the optional projection of X_t^+ on $\mathcal{H}_t$ is given as $\frac{1}{2}\alpha_t^+$, where (α_t) is the martingale in (4). Assume furthermore that $n_A(\mathrm{d}v)$ is *symmetric*. Recall that $K(L_t) - |B_t|k(L_t)$ is a martingale for any bounded Borel function k, where $K(y) = \int_0^y k(s)\mathrm{d}s$ (see [33]). This generalizes instantly, through the Dambis–Dubins–Schwarz theorem, to X in place of B. Thus for any bounded Borel function k, the process $M_t^k = K(L_t) - |X_t|k(L_t)$ is a martingale, when now L_t is the local time of X normalized so $(|X_t| - L_t)$ is a martingale. Since the local time (L_t) is adapted to $(\mathcal{H}_t)$, the projection of (M_t^k) on $(\mathcal{H}_t)$ is given as

$$m_t^k = K(L_t) - \frac{k(L_t)}{2n_A([t-g_t,\infty))} = K(L_t) - \frac{k(L_t)|\alpha_t|}{2}, \quad t \geq 0, \tag{19}$$

and is a $(\mathcal{H}_t)$-martingale.

Proposition 5. *Let X be a one-dimensional diffusion, martingale on (l, u), $l < 0 < u$ such that $n_A(\mathrm{d}v)$ is symmetric. Let μ be a probability measure on $(0,\infty)$ with $\int_0^\infty x\mathrm{d}\mu(x) < \infty$ and R be a stopping time such that $|\alpha_R| \sim \mu$ and $(\alpha_{t\wedge R} : t \geq 0)$ is a uniformly integrable martingale. Denote ρ_R the law of L_R. The following bound is true*

$$\mathbb{E}\left[\left(L_R - \overline{\rho}_R^{-1}(p)\right)^+\right] \leq \mathbb{E}\left[\left(L_{T^{|\alpha|}} - \overline{\rho}_{T^{|\alpha|}}^{-1}(p^*)\right)^+\right], \quad p \in [0,1], \tag{20}$$

where $T^{|\alpha|} = T^{|\alpha|}_{\varphi_\mu}$ is given in Theorem 4, the inverses $\overline{\rho}_\cdot^{-1}$ are taken left-continuous and $p^* = \overline{\mu}\left(\overline{\mu}^{-1}(p)\right) \geq p$.

The proof of Proposition 5 is just an application of the optional stopping theorem to the martingale in (19) with $k(l) = \mathbf{1}_{l \geq \overline{\rho}_R^{-1}(p)}$. We have

$$\begin{aligned}
\mathbb{E}\left[\left(L_R - \overline{\rho}_R^{-1}(p)\right)^+\right] &= \frac{1}{2}\mathbb{E}|\alpha_R|\mathbf{1}_{L_R \geq \overline{\rho}_R^{-1}(p)} \leq \frac{1}{2}\mathbb{E}|\alpha_R|\mathbf{1}_{|\alpha_R| \geq \overline{\mu}^{-1}(p)} \\
&= \frac{1}{2}\mathbb{E}|\alpha_{T^{|\alpha|}}|\mathbf{1}_{|\alpha_{T^{|\alpha|}}| \geq \overline{\mu}^{-1}(p)} \leq \frac{1}{2}\mathbb{E}|\alpha_{T^{|\alpha|}}|\mathbf{1}_{L_{T^{|\alpha|}} \geq \overline{\rho}_{T^{|\alpha|}}^{-1}(p^*)} \\
&= \mathbb{E}\left[\left(L_{T^{|\alpha|}} - \overline{\rho}_{T^{|\alpha|}}^{-1}(p^*)\right)^+\right],
\end{aligned}$$

which proves the Proposition. Note that in the statement we could just as well fix the law of $|A_R|$ as it is equivalent to fixing the law of $|\alpha_R| = 1/n_A([|A_R|,\infty))$, only the integrability condition on μ would change.

Proposition 5 and the bound of Obłój and Yor [36] can be summarized by saying that the law of L_∞, the local time of a continuous UI martingale (N_t), the distribution of $|N_\infty|$ or $|A_\infty|$ being fixed, is bounded in the *excess wealth order* and hence in the *convex order* (see [25]) and the upper bound is attained with stopping times of the type (18). This complements the study of Vallois [54] who obtained lower and upper bounds on the law of L_∞ in the *convex order*, under fixed distribution of N_∞ and showed that both bounds can be attained with solutions to the Skorokhod embedding problem for Brownian motion. The upper bound is attained with the solution developed by Vallois [53] which we recovered in Theorem 1. The lower bound was attained with an analogous solution presented by Vallois [54] which, in comparison with (9), takes the functions $\psi_{+/-}$ decreasing and not increasing. As noted after Theorem 1, we could also develop our general solution with decreasing functions $\psi_{+/-}$. This would lead to lower bounds on the law of L_∞ under fixed law of $|A_\infty|$.

4. Embeddings for Azéma's martingale and other signed functionals

Let X be a one-dimensional diffusion on (l, u), $l < 0 < u$. Recall the martingale $(\alpha_t : t \geq 0)$ displayed in (4) and the fact that the characteristic measure of the functional α is given by $n_\alpha(\mathrm{d}v) = \frac{\mathrm{d}v}{v^2}$, $v \neq 0$.

Proposition 6. *For a non-atomic measure μ on $\mathbb{R}$ there exists a stopping time S such that $\alpha_S \sim \mu$ and $(\alpha_{t\wedge S} : t \geq 0)$ is a uniformly integrable martingale if and only if the measure μ is centered, in which case we can take $S = T^\alpha_{\varphi_-,\varphi_+}$ defined via (9) and (13)–(15).*

Proof of Propositions 2, 3 and 6. We first prove Propositions 2 and 6 which are closely related. Suppose S is a stopping time such that $A_S \sim \mu$ and $(\alpha_{t\wedge S} : t \geq 0)$ is a uniformly integrable martingale. Then $\mathbb{E}|\alpha_S| < \infty$ and $\mathbb{E}\,\alpha_S = 0$. We have

$$D^A_\mu(y) = \int_0^y \frac{\mathrm{d}\mu(v)}{n_A([v,\infty))} = \mathbb{E}\frac{A_S \mathbf{1}_{0\leq A_S\leq y}}{n_A((A_S,\infty))} = \mathbb{E}\alpha_S \mathbf{1}_{0\leq A_S\leq y}, \tag{21}$$

where the second equality follows from the fact that $A_S \sim \mu$ and n_A is absolutely continuous with respect to the Lebesgue measure (cf. Section 2.2). In parallel with (21), we obtain $G^A_\mu(x) = \mathbb{E}\alpha_S \mathbf{1}_{x\leq A_S<0}$. We see that $\mathbb{E}\alpha_S = 0$ is equivalent to $D^A_\mu(\infty) = G^A_\mu(-\infty) < \infty$. However then the condition (12) is satisfied and Theorem 1 tells us that $T = T^A_{\varphi^A_-,\varphi^A_+} < \infty$ a.s. and $A_T \sim \mu$. Let T_n denote $T^A_{\varphi^A_-\wedge n,\varphi^A_+\wedge n}$. The process $(\alpha_{t\wedge T_n} : t \geq 0)$ is bounded and hence a uniformly integrable martingale. Furthermore, $T_n \to T$ so a sufficient condition for the uniform integrability of $(\alpha_{t\wedge T} : t \geq 0)$ is the uniform integrability of $(|\alpha_{T_n}| : n \geq 1)$. This in turn will follow from L^1 convergence of the sequence. We have $|\alpha_{T_n}| \to |\alpha_T|$ a.s. By Scheffé's lemma, it suffices to show $\mathbb{E}|\alpha_{T_n}| \to \mathbb{E}|\alpha_T|$, which follows from

$$\begin{aligned}\mathbb{E}|\alpha_{T_n}| &= \mathbb{E}|\alpha_T|\mathbf{1}_{|\alpha_T|<n} + n\mathbb{P}(|\alpha_T| \geq n)\\ &= \int_{-n}^n |x|\mathrm{d}\mu(x) + n\int_{|x|\geq n} \mathrm{d}\mu(x) \leq \int_{\mathbb{R}} |x|\mathrm{d}\mu(x) = \mathbb{E}|\alpha_T|.\end{aligned}$$

Thus, $(\alpha_{t\wedge T} : t \geq 0)$ is a uniformly integrable martingale if and only if α_T is integrable (and thus centered), which in turn is equivalent to $D^A_\mu(\infty) = G^A_\mu(-\infty) < \infty$. This ends the proof of Proposition 2. The proof of Proposition 6 is similar, it suffices to note that as $n_\alpha(\mathrm{d}x) = \frac{\mathrm{d}x}{x^2}\mathbf{1}_{x\neq 0}$, the condition $D^{\alpha}_{\mu}(\infty) = G^{\alpha}_{\mu}(-\infty) < \infty$ is equivalent to $\int_{\mathbb{R}} |x| \mathrm{d}\mu(x) < \infty$ and $\int_{\mathbb{R}} x \mathrm{d}\mu(x) = 0$. The uniform integrability of $(\alpha_{t\wedge T} : t \geq 0)$ is argued as above.

Likewise, the proof of Proposition 3 is immediate if we show that $n_M(\mathrm{d}y) = \frac{\mathrm{d}y}{2y^2}$. There are number of ways to see it. Here is one. From Corollary 2.1 and discussion in Section 2.4 in [42] it follows that $n_M((y, \infty))$ is proportional to the scale function $s(y)$ of the diffusion $(Y_t : t \geq 0)$ obtained upon conditioning X, starting at some point $x > 0$, to approach ∞ before 0. More precisely, we define Y as Doob's h-transform of X via $\mathbb{E}_x[H(Y_s : s \leq t)] = \mathbb{E}_x[\frac{X_{t\wedge T_0}}{x} H(X_s : s \leq t)]$, where H is a positive functional and $T_a = \inf\{t : X_t = a\}$. Taking $H(X_s : s \leq t) = \mathbf{1}_{t\wedge T_b < T_a}$, $0 < a < x < b$, using the fact that X is on a natural scale and letting $t \to \infty$, it is easy to verify that $s(y) = 1/y$ is a scale function for Y. Similar argument applies with $(-X)$ in place of X, which shows that n_M is symmetric. Thus $n_M(\mathrm{d}x)$ is proportional to $\mathrm{d}x/x^2$, $n_M(\mathrm{d}x) = c\mathrm{d}x/x^2$ for some positive constant c.

Note that in Proposition 3 we chose a specific normalization for the local time, described by the fact that $(L_t - |X_t|)$ is a local martingale. This allows us to recover c using the compensation formula (cf. [47, Prop. XII.2.6]). Let $\lambda > 0$ and $T_{-\lambda,\lambda} = \inf\{t : |X_t| \geq \lambda\}$, then we obtain

$$1 = \mathbb{E}\left[\sum_{g_s \leq T_{-\lambda,\lambda}} \mathbf{1}_{\sup_{g_s \leq u \leq d_s} X_u \geq \lambda} + \mathbf{1}_{\inf_{g_s \leq u \leq d_s} X_u \leq -\lambda}\right] = 2c \frac{\mathbb{E}L_{T_{-\lambda,\lambda}}}{\lambda} = 2c,$$

which yields $c = \frac{1}{2}$ and ends the proof. $\square$

We now specialize to the Brownian setup. Let $B = (B_t : t \geq 0)$ be a real-valued Brownian motion and $L = (L_t : t \geq 0)$ its local time at zero with $\mathbb{E}L_t = \mathbb{E}|B_t|$. Define functional p through $p(\epsilon, r) = \int_0^{r\wedge V(\epsilon)} \frac{\mathrm{d}s}{\epsilon(s)}$, which is F^1_{-1} in (6). We refer to the associated process $p_t = \int_{g_t}^t \frac{\mathrm{d}s}{B_s}$ as to the Brownian principal value. This is in fact an abuse of terminology as p_t is given as an absolutely convergent integral and it is its sum over excursions, the process $P_t = \int_0^t \frac{\mathrm{d}s}{B_s} = \lim_{x\to 0} \int_0^t \frac{\mathrm{d}s}{B_s} \mathbf{1}_{|B_s| \geq x}$, which needs to be understood as Cauchy's principal value (cf. [6]). Set $\tilde{p} = \frac{1}{2} p$. Introduce also $\tilde{\alpha}_t = \sqrt{\frac{\pi}{2}} \mathrm{sgn}(B_t) \sqrt{t - g_t}$ the Azéma martingale and recall that it is the projection of B on the filtration generated by its zeros.

Proposition 7. *Let μ be a non-atomic probability measure on $\mathbb{R}$, such that $\int_{\mathbb{R}_+} |s| \mathrm{d}\mu(s) = \int_{\mathbb{R}_-} |s| \mathrm{d}\mu(s)$. Define $\psi_{+/-}, \varphi_{+/-}$ via (13)–(15). Then we have $B_{T^M} \sim \tilde{p}_{T^{\tilde{p}}} \sim \tilde{\alpha}_{T^{\tilde{\alpha}}} \sim \mu$. Furthermore, $T^M, T^{\tilde{p}}$ and $T^{\tilde{\alpha}}$ are a.s. finite stopping times in the natural filtration of (B_t), $(\tilde{p}_t)$ and $(\tilde{\alpha}_t)$ respectively. The martingales $(B_{t\wedge T^M} : t \geq 0)$ and $(\tilde{\alpha}_{t\wedge T^{\tilde{\alpha}}} : t \geq 0)$ are uniformly integrable if and only if $\int_{\mathbb{R}} |x| \mathrm{d}\mu(x) < \infty$, in which case μ is centered.*

Proof. We have $X = B$ a Brownian motion. Recall the notation of Section 3. As φ_+ and φ_- are increasing, we have $M_{T^M_{\varphi_-,\varphi_+}} = B_{T^M_{\varphi_-,\varphi_+}}$, where $M_t = \mathrm{sgn}(B_t) \sup_{g_t \leq s \leq t} |B_s|$ is associated with the functional M given by (7). We will now show that

$$n_M(\mathrm{d}x) = n_{\tilde{\alpha}}(\mathrm{d}x) = n_{\tilde{p}}(\mathrm{d}x) = \frac{\mathrm{d}x}{x^2} \mathbf{1}_{x\neq 0}. \tag{22}$$

This will end the proof, as then we can proceed exactly as in the proof of Propositions 2 and 6 above. Here we can also exploit the well known fact that for $\mathbb{E}|B_T| < \infty$ the conditions: T is minimal and $(B_{t\wedge T} : t \geq 0)$ is a uniformly integrable martingale, are equivalent (cf. [32, Sec. 8]).

The assertion for $n_{\tilde{\alpha}}$ is a direct consequence of the independence between the length and the sign of an excursion and the expression of the characteristic measure of the age functional, $V(\epsilon)(t) = t \wedge V(\epsilon)$, given by $n_V = \frac{\mathrm{d}v}{\sqrt{2\pi v^3}}$ (cf. [47, Prop. XII.2.8]).

The assertion (22) for n_M is well known (cf. [47, Prop. XII.3.6]) and we argued it in a greater generality above in the proof of Proposition 6.

Finally, the assertion on the characteristic measure of $\tilde{p}$ follows readily from Theorem 4.1 in [6], but we present another simple justification. We look at the process of terminal values of p, $p_{\tau_l-} = \int_0^{\tau_l - \tau_{l-}} \frac{\mathrm{d}s}{B_{\tau_{l-}+s}}$. It follows easily from the Poisson point process properties of the excursion process, that $H_l = P_{\tau_l} = \sum_{u \le l} p_{\tau_u -}$ is a Lévy process. Examining the scaling property for H one finds that $\frac{1}{\pi} H_l$ is actually a standard Cauchy process. Now using the exponential formula we can calculate $\mathbb{E}\mathrm{e}^{\mathrm{i}\lambda p_t}$ and comparing it with the known quantity for Cauchy process, we find $n_p(\mathrm{d}x) = \frac{\mathrm{d}x}{x^2}\mathbf{1}_{x \neq 0}$ and thus $n_{\tilde{p}}(\mathrm{d}x) = \frac{\mathrm{d}x}{2x^2}$, $x \neq 0$. $\square$

Note that the above proposition is quite remarkable as we have actually the same formula for the stopping time which works both for Brownian motion and for its projection on the filtration generated by the signs that is the Azéma martingale.

The last process we want to mention here is the skew Brownian motion used recently by Cox and Hobson [14] to develop a class of embeddings in Brownian motion. Intuitively speaking, it is a Brownian motion which chooses positive excursions with probability p, $0 < p < 1$, and negative excursions with probability $(1 - p)$. Naturally, we know the characteristic measure of the signed extrema functional (7), $n_M(\mathrm{d}x) = \frac{p\mathrm{d}x}{x^2}\mathbf{1}_{x>0} + \frac{(1-p)\mathrm{d}x}{x^2}\mathbf{1}_{x<0}$, and we can thus develop an explicit solution to the Skorokhod embedding problem for skewed Brownian motion.

We present two explicit calculations of functions $\varphi_{+/-}$ in the case $n_F(\mathrm{d}x) = \mathrm{d}x/x^2$, $x \neq 0$, which we encountered in Propositions 3, 6 and 7. We look only on asymmetric probability measures as the symmetric case follows immediately from the solution for positive functionals (cf. Section 5 below).

Double exponential. Let $\mu(\mathrm{d}x) = \frac{\lambda^2}{\lambda+\gamma}\mathrm{e}^{-\lambda x}\mathbf{1}_{x>0} + \frac{\gamma^2}{\lambda+\gamma}\mathrm{e}^{\gamma x}\mathbf{1}_{x<0}$, for some $\lambda, \gamma > 0$. The coefficients are chosen so that μ is a centered probability measure. We have $D_\mu(y) = (1 - \mathrm{e}^{-\lambda y}(1 + \lambda y))/(\lambda + \gamma)$, $y \ge 0$, and $G_\mu(x) = (1 - \mathrm{e}^{\gamma x}(1 - \gamma x))/(\lambda + \gamma)$, $x \le 0$. Note that $D_\mu(\infty) = 1/(\lambda + \gamma) = G_\mu(-\infty)$ so that (12) is indeed verified. We see easily that $G_\mu(\lambda x/\gamma) = D_\mu(-x)$ and $D_\mu(\gamma y/\lambda) = G_\mu(-y)$. This yields

$$\varphi_-(y) = \frac{1}{\lambda}\sqrt{(\lambda+\gamma)y} \quad \text{and} \quad \varphi_+(y) = \frac{1}{\gamma}\sqrt{(\lambda+\gamma)y}. \tag{23}$$

F-uniform. Let $\mu(\mathrm{d}x) = K(\mathbf{1}_{x \ge g}/x^2 + \mathbf{1}_{x \le -h}/x^2)\mathrm{d}x$ where $g, h > 0$ and $K(1/g + 1/h) = 1$. We called this measure *F-uniform* as it is just a (weighted) restriction of the measure $n_F(\mathrm{d}x)$ to $\mathbb{R} \setminus (-h, g)$. In particular it is easy to justify that $\varphi_{+/-}$ have to be affine functions. We recover the formulae presented in [53]. We have $D_\mu(y) = K\log(y/g)$, $y \ge g$, and $G_\mu(x) = K\log(-x/h)$, $x < -h$. Note that $G_\mu(-\infty) = D_\mu(\infty) = \infty$ so (12) is verified. We obtain

$$\varphi_+(y) = g\left(1 + \frac{y}{2K}\right) \quad \text{and} \quad \varphi_-(y) = h\left(1 + \frac{y}{2K}\right). \tag{24}$$

5. Embeddings for Bessel and Cox–Ingersoll–Ross processes

In this section, we apply Theorem 4 to obtain embeddings for positive functionals. More precisely, we specialize to the case of F with $n_F(\mathrm{d}x) = \frac{\mathrm{d}x}{x^2}\mathbf{1}_{x>0}$, which was studied in [35]. We have $n_F([y,\infty)) = 1/y$ and so we obtain

$$\psi_\mu(t) = -\int_{[0,t)} s\mathrm{d}(\ln \overline{\mu}(s)) = \int_0^t \frac{s\mathbf{1}_{\mu(\{s\})=0}\mathrm{d}\mu(s)}{\overline{\mu}(s)} + \sum_{s\le y} s\ln\left(\frac{\overline{\mu}(s)}{\overline{\mu}(s+)}\right)\mathbf{1}_{\mu(\{s\})>0}, \quad (25)$$

which is the correct definition of the *dual Hardy–Littlewood function*, introduced in [35, Eq. 3.1].

Let $(R_t^{(q)} : t \ge 0)$ be a Bessel process with index $q \in (-1, 0)$, starting in zero (we write $\mathrm{BES}^{(q)}$). For background on Bessel processes we refer to [47, Ch. XI]. The processes $R^{(q)}$ for $q \in (-1,0)$ are nice Markov processes as in Section 2. We exploit here the description of their Itô measure n^q, due to [41] (see also [6, p. 43]), which provides an important generalization of Williams' decomposition of Brownian excursions (cf. [55, Sec. 67], [48]). We choose the normalization of Itô's measure and the local time under which the process $(R_t^{(q)})^{-2q} - L_t$ is a martingale (see [17] for a survey of common normalizations). We recall however that our results are independent of renormalization of the local time and Itô's measure.

Consider two functionals of the excursions of $R^{(q)}$, given via

$$\tilde{M}(\epsilon)(r) = \left(\sup_{s\le r\wedge V(\epsilon)} \epsilon(s)\right)^{2|q|} \quad \text{and} \quad \tilde{A}(\epsilon)(r) = (r\wedge V(\epsilon))^{|q|}. \quad (26)$$

It follows from [41] that their characteristic measures are given by $n_{\tilde{M}}(\mathrm{d}x) = \frac{\mathrm{d}x}{x^2}\mathbf{1}_{x>0}$ and $n_{\tilde{A}}(\mathrm{d}v) = \frac{c\mathrm{d}v}{x^2}\mathbf{1}_{v>0}$, where $c = \frac{2^q}{\Gamma(|q|+1)}$.

We can apply Theorem 4 generalizing the results of Obłój and Yor [35] from Brownian motion to any Bessel process with index $q \in (-1,0)$. We note that the same result was obtained independently by Nikeghbali [31] using entirely different methods. For simplicity we treat measures without an atom in zero.

Proposition 8. *Let μ be a probability measure on $\mathbb{R}_+$ with $\mu(\{0\}) = 0$. Define the dual Hardy–Littlewood function ψ_μ through (25) and let φ_μ denote its right-continuous inverse. Then the stopping times*

$$\begin{aligned} T^R &= \inf\{t>0 : \tilde{M}_t \ge \varphi_\mu(L_t)\} = \inf\{t>0 : R_t^{(q)} \ge \varphi_\mu(L_t)^{1/2|q|}\} \\ &= \inf\{t>0 : \sup_{s\le t} R_s^{(q)} \ge \varphi_\mu(L_t)^{1/2|q|}\}, \quad (27) \\ T^{\tilde{A}} &= \inf\{t>0 : \tilde{A}_t \ge \varphi_\mu(cL_t)\} = \inf\{t>0 : (t-g_t) \ge \varphi_\mu(cL_t)^{1/|q|}\}, \end{aligned}$$

solve the Skorokhod embedding problem, i.e. $\tilde{M}_{T^R} = (\sup_{s\le T^R} R_s^{(q)})^{2|q|} = \left(R_{T^R}^{(q)}\right)^{2|q|} \sim \mu$ *and* $\tilde{A}_{T^{\tilde{A}}} \sim \mu$, *where* $q \in (-1,0)$ *and* $c = 2^q/\Gamma(|q|+1)$.

Proof. The embedding for $\tilde{M}$ is a simple application of Theorem 4. As φ_μ is increasing it is straightforward to see that $\tilde{M}_{T_R} = (\sup_{s\le T^R} R_s^{(q)})^{2|q|}$, and as the local time L_t is constant on excursions away from zero, T^R is a point of increase for the maximum process of $R^{(q)}$ and thus $R_{T_R}^{(q)} = \sup_{s\le T^R} R_s^{(q)}$.

To prove the embedding for $\tilde{A}$ is suffices to notice that the function $\psi_\mu^{\tilde{A}}$ associated to $\tilde{A}$ by (16) is linked with the *dual Hardy–Littlewood* function (25) through $\psi_\mu^{\tilde{A}} = \frac{1}{c}\psi_\mu$, and thus their inverses satisfy $\varphi_\mu^{\tilde{A}}(y) = \varphi_\mu(cy)$. $\square$

The discussion above can be extended to Bessel processes with drift downwards. Donati-Martin and Yor [18] gave a formula which expresses the characteristic measure of the age functional of $\mathrm{BES}(q, \beta\downarrow)$ process, that is a downwards $\mathrm{BES}^{(-q)}$ process with "drift" β (cf. [40]).

We turn now to an embedding for the age process of excursions for the Cox–Ingersoll–Ross processes. Fix $\gamma > 0$, $\delta \in (0, 2)$ and let $(X_t : t \geq 0)$ be the non-negative solution of

$$\mathrm{d}X_t = 2\sqrt{X_t}\mathrm{d}B_t + (\delta - 2\gamma X_t)\mathrm{d}t, \tag{28}$$

where we assume $X_0 = 0$, and where (B_t) is a real-valued Brownian motion. The processes $X = X^{\delta,\gamma}$ found applications in mathematical finance (cf. [57, Ch. 5]) and are called the *Cox–Ingersoll–Ross* processes. They are also called the *squared Ornstein–Uhlenbeck process* with dimension δ (cf. [43]) and actually date back to the work of Motoo [30]. Denote $\Lambda = \Lambda^{\delta,\gamma}$ the Lévy measure of τ, which is the characteristic measure of the age functional A given in (3), $\Lambda((v,\infty)) = n_A((v,\infty))$. This measure is known and given by

$$\Lambda^{\delta,\gamma}((v,\infty)) = C\frac{\mathrm{e}^{-2\gamma(1-\frac{\delta}{2})v}}{(1-\mathrm{e}^{-2\gamma v})^{(1-\frac{\delta}{2})}}, \tag{29}$$

where C is a constant which depends on the normalization of the local time L (cf. [43, Eq. 59]). Theorem 4 thus gives instantly an explicit solution to the Skorokhod embedding problem for age process $A_t = t - g_t$ of the Cox–Ingersoll–Ross processes.

Examples of *dual Hardy–Littlewood* functions for several measures were already given in [35] so we report here only the case of geometric law which had a mistake in [35]. Let μ be a probability measure on $\mathbb{N}$ with $\mu(\{k\}) = (1-p)^{k-1}p$, for certain $0 < p < 1$, $k \in \mathbb{N}$. Then $\overline{\mu}(k) = (1-p)^{k-1}$ and

$$\psi_\mu(k) = \frac{k(k+1)}{2}\log\left(\frac{1}{1-p}\right)$$
$$\varphi_\mu(x) = k \quad \text{for } \frac{(k-1)k}{2}\log\left(\frac{1}{1-p}\right) \leq x < \frac{k(k+1)}{2}\log\left(\frac{1}{1-p}\right).$$

More generally, for any discrete probability measure we have $\varphi_\mu(x) = k$ for $x \in [a_{k-1}, a_k)$, where $a_k = \psi_\mu(k)$.

6. Atomic measures in the signed case

So far, for signed functionals, we have only considered non-atomic measures. In the presence of atoms we do not obtain closed-form formulae and the functions $\psi_{+/-}$ are given through an iteration procedure. We will not phrase the result precisely but only explain the procedure and present an example.

Let F be a functional as described in Section 2.2 and μ a purely atomic probability measure, $\mu = \sum_{k=1}^n a_k\delta_{x_k} + \sum_{k=1}^m b_k\delta_{y_k}$, where $x_{k+1} < x_k < 0$, $y_{k+1} > y_k > 0$ and $a_k, b_k > 0$, $\sum_{k=1}^n a_k + \sum_{k=1}^m b_k = 1$, $n, m \leq \infty$. Suppose that $n_F((-\infty, x_n]) > 0$ and $n_F([y_m,\infty)) > 0$. We will describe functions φ_+ and φ_- such that the stopping time $T^F_{\varphi_-,\varphi_+}$ given by (9) solves the Skorokhod embedding problem for F, i.e. $F_{T^F_{\varphi_-,\varphi_+}} \sim \mu$.

Naturally $\varphi_- : \mathbb{R}_+ \to \{-x_1, \dots, -x_n\}$, $\varphi_+ : \mathbb{R}_+ \to \{y_1, \dots, y_m\}$ and $\varphi_{+/-}$ are piece-wise constant and non-decreasing. We thus have

$$\varphi_-(z) = -x_k \quad \alpha_{k-1} \leq z < \alpha_k; \qquad \varphi_+(y) = y_k \quad \beta_{k-1} \leq y < \beta_k, \tag{30}$$

for some positive, finite, increasing sequences $(\alpha_k : 1 \leq k < n)$ and $(\beta_k : 1 \leq k < m)$, and $\alpha_0 = \beta_0 = 0, \alpha_n = \beta_m = \infty$. Our aim is to determine the values α_k, β_k in terms of n_F and μ. For ease of notation we write $n_F(x) = n_F((-\infty, x])$ and $\overline{n}_F(y) = n_F([y, \infty))$ and $T = T^F_{\varphi_-,\varphi_+}$.

The formula in (38) is still valid and taking $h(z) = \mathbf{1}_{z=x_i}$ and $h(z) = \mathbf{1}_{z=y_j}$, we find

$$a_i = n_F(x_i) \int_{\alpha_{i-1}}^{\alpha_i} \mathbb{P}(L_T \geq l)\mathrm{d}l, \qquad b_k = \overline{n}_F(y_j) \int_{\beta_{j-1}}^{\beta_j} \mathbb{P}(L_T \geq l)\mathrm{d}l \tag{31}$$

$1 \leq i \leq n, 1 \leq j \leq m$, where $\mathbb{P}(L_T \geq l)$ is given through (33). One can check that (31) identifies the functions $\varphi_{+/-}$ uniquely. It also encodes the conditions on μ imposing $\alpha_i, \beta_j < \infty, i < n, j < m$ and $\alpha_n = \beta_m = \infty$.

Example. Let $n_F(\mathrm{d}x) = \mathrm{d}x/x^2, x \neq 0$ and $\mu = a\delta_{x_1} + b\delta_{y_1} + (1 - a - b)\delta_{y_2}$ with $a, b > 0, x_1 < 0 < y_1 < y_2$. With the notation used above $n = 1, m = 2$ so that $\alpha_1 = \infty = \beta_2$. Thus, we need to have $\beta_1 < \alpha_1$ which happens if $by_1 < a|x_1|$, which is the first condition we impose on μ. We then proceed to calculate β_1 relying on (31). We find

$$\beta_1 = \frac{|x_1|y_1}{y_1 - x_1} \log\left(\frac{x_1}{b(y_1 - x_1) - x_1}\right). \tag{32}$$

To end the construction we have to require that (31) yields $\alpha_1 = \beta_2$ which is equivalent to $by_1 + (1 - a - b)y_2 = ax_1$, that is to say μ is centered.

7. Proofs of the main results

In this last section we present the proofs of Theorems 1 and 4. We start with the former which is more technical and parts of which are then used in the proof of the latter. However, first we need to argue that the objects we introduced are well defined. Indeed, recall that the excursion measure n and the local time L are uniquely determined only up to a constant multiplicative factor. We need to argue that our results are independent of renormalization of n and L.

Suppose that we multiply the excursion measure by some constant c. The change $n \rightsquigarrow c \cdot n$ is directly translated into $n_F \rightsquigarrow c \cdot n_F$, which in turn gives $G_\mu \rightsquigarrow \frac{1}{c}G_\mu$ and $D_\mu \rightsquigarrow \frac{1}{c}D_\mu$. Such a transformation leaves unchanged both $D_\mu^{-1}(G_\mu(\cdot))$ and $G_\mu^{-1}(D_\mu(\cdot))$, thus leading to the change $\psi_+ \rightsquigarrow \frac{1}{c}\psi_+$ and $\psi_- \rightsquigarrow \frac{1}{c}\psi_-$, which in turn translates into $\varphi_+(\cdot) \rightsquigarrow \varphi_+(c\cdot)$ and $\varphi_-(\cdot) \rightsquigarrow \varphi_-(c\cdot)$. Multiplying the excursion measure n by a constant c induces a multiplication of the local time at zero by the constant $\frac{1}{c}$, as can be readily seen from the relationship between the Lévy measure of τ and the measure n (cf. [4, Lemma IV.9]), or from the fact that the measure $\mathrm{d}n(\epsilon)\mathrm{d}L_s$ is an invariant of excursion theory (which in turn follows easily from the compensation formula). In consequence, the quantities $\varphi_+(L_t)$ and $\varphi_-(L_t)$ both stay unchanged, which implies that the stopping times given in (9) stay unchanged as well.

7.1. *Proof of the embedding for signed functionals*

We prove Theorem 1. Instead of just verifying that our embedding works we choose to present rather the complete reasoning which allows us to obtain our solution.

We start by calculating the law of L_T. For ease of notation, we denote the terminal value $F(\epsilon, V(\epsilon))$ simply by $F(\epsilon)$. We have

$$\begin{aligned}\mathbb{P}\left(L_{T^F_{\varphi_-,\varphi_+}} > l\right) &= \mathbb{P}\left(T^F_{\varphi_-,\varphi_+} > \tau_l\right)\\ &= \mathbb{P}\,(\text{on the time interval } [0,\tau_l] \text{ for every excursion } e_s,\ s \le l,\\ &\quad\ \text{the values of } F \text{ were between } -\varphi_-(s) \text{ and } \varphi_+(s))\\ &= \mathbb{P}\,(\forall s \le l,\ F(e_s, V(e_s)) \in (-\varphi_-(s), \varphi_+(s)))\\ &= \exp\left(-\int_0^l n_F\,((-\infty, -\varphi_-(s)] \cup [\varphi_+(s), +\infty))\,\mathrm{d}s\right). \end{aligned} \tag{33}$$

We note that the law of $L_{T^F_{\varphi_-,\varphi_+}}$ is absolutely continuous with respect to the Lebesgue measure. As $L_\infty = \infty$ a.s., the above gives us a convenient criterion for finiteness of our stopping time, namely $T^F_{\varphi_-,\varphi_+} < \infty$ a.s. if and only if the integral in (33), with $l = \infty$, is infinite. We now prove the latter.

Recall that we assumed that $n_F((-\infty, x]) > 0$ and $n_F([y, \infty)) > 0$ for $a_\mu < x \le 0 \le y < b_\mu$, and that (12) holds. This ensures that the functions ψ_+ and ψ_-, given via (13) and (14), are well defined. Let $\lambda = \psi_+(b_\mu) = \psi_+(\infty)$. We have $\psi_-(\infty) = \psi_-(-a_\mu) = \psi_+(D_\mu^{-1}(G_\mu(a_\mu))) = \psi_+(b_\mu) = \lambda$, where we used the assumption (12) that $D_\mu(b_\mu) = G_\mu(a_\mu)$. We denote this last value by $c_\mu = D_\mu(b_\mu)$. We have

$$\begin{aligned}\int_0^\infty n_F([\varphi_+(s), \infty))\mathrm{d}s &= \int_0^{b_\mu} \frac{\mathrm{d}\mu(s)}{1 + \overline{\mu}(s) - \overline{\mu}(G_\mu^{-1}(D_\mu(s)))}\\ &= \int_0^{c_\mu} \frac{n_F([D_\mu^{-1}(v), \infty))\mathrm{d}v}{1 + \overline{\mu}(D_\mu^{-1}(v)) - \overline{\mu}(G_\mu^{-1}(v))},\end{aligned} \tag{34}$$

where the equalities follow with a change of variables from (13) and (10). This is easy when μ has a positive density but is also true in the general setting. Indeed, since μ has no atoms, ψ_+ and ψ_- are continuous and $\psi_{+/-}(\varphi_{+/-}(y)) = y$. Jumps of $\varphi_{+/-}$ correspond to the level stretches of $\psi_{+/-}$, so that $\mathrm{d}\psi_{+/-}$ – a.e. $\varphi_{+/-}(\psi_{+/-}(y)) = y$. This justifies the first equality in (34). For the second one, note that the functions D_μ and G_μ are constant only outside of the support of μ, so that any $y > 0$, $y \in \mathrm{supp}(\mu)$, can be represented as $D_\mu^{-1}(u)$ and any $x < 0$, $x \in \mathrm{supp}(\mu)$, can be represented as $G_\mu^{-1}(v)$, for some u, v. These remarks justify also the following derivation, based on (14) and (10)

$$\int_0^\infty n_F((-\infty, -\varphi_-(s)])\mathrm{d}s = \int_0^{c_\mu} \frac{n_F((-\infty, G_\mu^{-1}(v)])\mathrm{d}v}{1 + \overline{\mu}(D_\mu^{-1}(v)) - \overline{\mu}(G_\mu^{-1}(v))}. \tag{35}$$

Now observe that

$$\begin{aligned}&\mathrm{d}(\overline{\mu}(D_\mu^{-1}(v))) = -n_F([D_\mu^{-1}(v), \infty))\mathrm{d}v \quad \text{and}\\ &\mathrm{d}(\overline{\mu}(G_\mu^{-1}(v))) = n_F((-\infty, G_\mu^{-1}(v)])\mathrm{d}v.\end{aligned} \tag{36}$$

This allows us to calculate the desired integral:

$$\begin{aligned}&\int_0^\infty n_F\,((-\infty, -\varphi_-(s)] \cup [\varphi_+(s), +\infty))\,\mathrm{d}s \quad \text{(using (34) and (35))}\\ &= \int_0^{c_\mu} \frac{n_F\left((-\infty, G_\mu^{-1}(v)] \cup [D_\mu^{-1}(v), \infty)\right)}{1 + \overline{\mu}(D_\mu^{-1}(v)) - \overline{\mu}(G_\mu^{-1}(v))}\mathrm{d}v \quad \text{(using (36))}\\ &= -\log\,(1 + \overline{\mu}(\infty) - \overline{\mu}(-\infty)) = +\infty,\end{aligned} \tag{37}$$

where we used the fact that $\overline{\mu}(D_\mu^{-1}(0)) = \overline{\mu}(G_\mu^{-1}(0))$ and $D_\mu^{-1}(c_\mu) = \infty$, $G_\mu^{-1}(c_\mu) = -\infty$. We proved, by (33), that $L_{T^F_{\varphi_-,\varphi_+}} < \infty$ a.s. and thus that $T^F_{\varphi_-,\varphi_+} < \infty$ a.s. From (37) above, it can also be deduced that $\psi_+(b_\mu) + \psi_-(-a_\mu) = \infty$ and thus, as $\psi_+(b_\mu) = \psi_-(-a_\mu)$, we see that both are infinite.

We now turn to the proof of the embedding property announced in Theorem 1. The terminal value for F_t, for a given excursion, is either achieved on some interval or not achieved at all and we deduce instantly that $F_T \in \{-\varphi_-(L_T), \varphi_+(L_T)\}$. From the properties of Poisson point processes, we see that conditionally on $\{L_T = l\}$, the respective probabilities that $F_T = -\varphi_-(L_T)$ or that $F_T = \varphi_+(L_T)$, are given by the proportions of the characteristic measures of appropriate regions, thus

$$\mathbb{P}(F_T = -\varphi_-(L_T)|L_T = l) = \frac{n_F((-\infty, -\varphi_-(l)])}{n_F((-\infty, -\varphi_-(l)] \cup [\varphi_+(l), +\infty))}.$$

Let $h : \mathbb{R} \to \mathbb{R}_+$ be a bounded Borel function. We can then write

$$\begin{aligned}
\mathbb{E}h(F_T) = \mathbb{E}\left(\mathbb{E}\left[h(F_T)|L_T\right]\right) &= \mathbb{E}\Big(\mathbb{E}\Big[h(-\varphi_-(L_T))\mathbf{1}_{F_T=-\varphi_-(L_T)} \\
&\quad + h(\varphi_+(L_T))\mathbf{1}_{F_T=\varphi_+(L_T)}|L_T\Big]\Big) \\
&= \int_0^\infty \mathbb{P}(L_T \in \mathrm{d}l)\left[\frac{h(-\varphi_-(l))n_F((-\infty, -\varphi_-(l)])}{n_F((-\infty, -\varphi_-(l)] \cup [\varphi_+(l), +\infty))}\right. \\
&\quad \left. + \frac{h(\varphi_+(l))n_F([\varphi_+(l), +\infty))}{n_F((-\infty, -\varphi_-(l)] \cup [\varphi_+(l), +\infty))}\right]. \qquad (38)
\end{aligned}$$

The above formula, in a Brownian setup and for the signed extrema functional (7), was obtained by Jeulin and Yor [22] (cf. [53, Eq. (2.3)]).

On the other hand, since we want to have $F_T \sim \mu$, the above displayed (38) has to be equal to $\int_{\mathbb{R}} h(x)\mathrm{d}\mu(x)$. This has to be true for any bounded function h and we will see that it will allow us to determine the functions φ_- and φ_+. Write ψ_- and ψ_+ respectively for the inverses of φ_- and φ_+, and assume that ψ_+, ψ_- are continuous (recall that this is indeed our case since the measure μ in Theorem 1 does not have any atoms). Fix $y > 0$ and put $h(z) = \mathbf{1}_{z\geq y}$. This yields

$$\begin{aligned}
\overline{\mu}(y) &= \int_{\psi_+(y)}^\infty \mathbb{P}(L_T \in \mathrm{d}l)\left[\frac{n_F([\varphi_+(l), +\infty))}{n_F((-\infty, -\varphi_-(l)] \cup [\varphi_+(l), +\infty))}\right] \\
&= \int_{\psi_+(y)}^\infty \mathrm{d}l\; n_F([\varphi_+(l), +\infty))\mathbb{P}(L_T \geq l), \qquad (39)
\end{aligned}$$

where we differentiated (33) to obtain $\mathbb{P}(L_T \in \mathrm{d}l)$. Similarly, if we fix $x < 0$ and put $h(z) = \mathbf{1}_{z\geq x}$, we obtain

$$\overline{\mu}(x) = \mathbb{P}\left(L_T \leq \psi_-(-x)\right) + \int_{\psi_-(-x)}^\infty \mathrm{d}l\; n_F([\varphi_+(l), +\infty))\mathbb{P}(L_T \geq l), \qquad (40)$$

where we used the assumption that ψ_- is continuous. Assume that $f : \mathbb{R}_- \to \mathbb{R}_+$ and $g : \mathbb{R}_+ \to \mathbb{R}_-$, given by

$$f(x) = \varphi_+(\psi_-(-x)) \quad \text{and} \quad g(y) = -\varphi_-(\psi_+(y)) \qquad (41)$$

are well defined and finite for $a_\mu < x \leq 0 \leq y < b_\mu$.

Recall the remarks between (34) and (35). In particular note that we can assume that $f(g(y)) = y$, $\mathrm{d}\psi_+(y)$ – a.e., and $g(f(x)) = x$, $\mathrm{d}\psi_-(-x)$ – a.e.

Let $y > 0$ and differentiate (39) to obtain

$$\mathrm{d}\overline{\mu}(y) = -n_F([y, +\infty))\mathbb{P}(L_T \geq \psi_+(y))\mathrm{d}\psi_+(y). \tag{42}$$

Now it suffices to note that

$$\begin{aligned}\mathbb{P}(L_T < \psi_+(y)) &= \mathbb{P}(L_T < \psi_-(-g(y))), \quad \text{by (40)}\\ &= \overline{\mu}(g(y)) - \int_{\psi_+(y)}^{\infty} \mathrm{d}l\, n_F([\varphi_+(l), \infty))\mathbb{P}(L_T \geq l)\\ &= \overline{\mu}(g(y)) - \overline{\mu}(y), \quad \text{using (39)}.\end{aligned} \tag{43}$$

Combining (42) and (43) above, we conclude that

$$\mathrm{d}\psi_+(y) = \frac{-\mathrm{d}\overline{\mu}(y)}{n_F([y, \infty))(1 + \overline{\mu}(y) - \overline{\mu}(g(y)))}. \tag{44}$$

We will try to obtain a similar expression starting with (40) instead of (39). To this end fix $x < 0$ and rewrite (40) using (39) in the following way

$$\overline{\mu}(x) = 1 - \mathbb{P}(L_T \geq \psi_-(-x)) + \overline{\mu}(f(x)), \tag{45}$$

where we used the fact that $\psi_+(f(x)) = \psi_-(-x)$. Note that we can assume that $\overline{\mu}(f(x))$ is continuous. Differentiating (45), through a reasoning similar to (43), we obtain

$$\begin{aligned}\mathrm{d}\overline{\mu}(x) &= (n_F((-\infty, x]) + n_F([f(x), +\infty)))\\ &\quad \times (1 + \overline{\mu}(f(x)) - \overline{\mu}(x))\,\mathrm{d}\psi_-(-x) + \mathrm{d}\overline{\mu}(f(x)),\end{aligned} \tag{46}$$

where we used the fact that $\mathrm{d}\psi_- -$ a.e. $\varphi_-(\psi_-(x)) = -x$. Taking $x = g(y)$, $y \geq 0$, in the above, yields

$$\mathrm{d}\psi_+(y) = \frac{\mathrm{d}\overline{\mu}(g(y)) - \mathrm{d}\overline{\mu}(y)}{(n_F((-\infty, g(y)]) + n_F([y, +\infty)))(1 + \overline{\mu}(y) - \overline{\mu}(g(y)))}. \tag{47}$$

Comparing (44) with (47) and simplifying the common terms we obtain finally

$$\frac{\mathrm{d}\overline{\mu}(y)}{n_F([y, +\infty))} = -\frac{\mathrm{d}\overline{\mu}(g(y))}{n_F((-\infty, g(y)])}, \quad y \geq 0. \tag{48}$$

It is therefore natural to introduce the functions D_μ and G_μ, defined in (10), which we recall here

$$D_\mu(y) = \int_0^y \frac{\mathrm{d}\mu(s)}{n_F([s, +\infty))} \quad \text{and} \quad G_\mu(x) = \int_x^0 \frac{\mathrm{d}\mu(s)}{n_F((-\infty, s])}, \tag{49}$$

for $y \geq 0$ and $x \leq 0$, and their right-continuous inverses denoted D_μ^{-1} and G_μ^{-1} respectively. The equality (48) reads $D_\mu(y) = G_\mu(g(y))$ or equivalently

$$f(x) = D_\mu^{-1}(G_\mu(x)), \quad x \leq 0, \quad \text{and} \quad g(y) = G_\mu^{-1}(D_\mu(y)), \quad y \geq 0. \tag{50}$$

Functions f and g were supposed to be well defined and we can now translate this assumption into conditions on F and μ. Namely, we need to have $D_\mu(y) < \infty$, $G_\mu(x) < \infty$ for $x < 0 < y$ and $D_\mu(\infty) = G_\mu(\infty)$. The first condition means that for x, y in the support of μ we need to have $n_F((-\infty, x]) > 0$ and $n_F([y, \infty)) > 0$, which we assumed in (11) and the second one is just (12).

We are finally able to justify the explicit formulae for ψ_+ and ψ_-. Indeed, substituting the expression (50) for g in (44) and passing to the integral representation, we obtain (13), that is for $y \geq 0$,

$$\psi_+(y) = \int_0^y \frac{\mathrm{d}\mu(s)}{n_F\,([s,+\infty))\,(1+\overline{\mu}(s)-\overline{\mu}(G_\mu^{-1}(D_\mu(s))))}, \tag{51}$$

and the formula for ψ_- follows from (48) and (50) as $\psi_-(y) = \psi_+(f(-y))$.

It remains to prove that T is minimal. Suppose that $S \leq T$ is a stopping time with $F_S \sim F_T$. From the definition of T in (9) it follows that $0 < F_S \leq \varphi_+(L_S) \leq \varphi_+(L_T)$ or $0 \geq F_S \geq -\varphi_-(L_S) \geq -\varphi_-(L_T)$. We obtain for $\lambda, \gamma > 0$

$$\begin{aligned} \overline{\mu}(\lambda) &= \mathbb{E}\mathbf{1}_{F_S\geq\lambda} \leq \mathbb{E}\left[\mathbf{1}_{F_S\geq 0}\mathbf{1}_{\varphi_+(L_S)\geq\lambda}\right] \leq \mathbb{E}\left[\mathbf{1}_{F_S\geq 0}\mathbf{1}_{\varphi_+(L_T)\geq\lambda}\right] \\ &= \overline{\mu}(\lambda) + \mathbb{E}\left[\left(\mathbf{1}_{F_S\geq 0}-\mathbf{1}_{F_T\geq 0}\right)\mathbf{1}_{\varphi_+(L_T)\geq\lambda}\right] = \overline{\mu}(\lambda)+\xi_\lambda, \end{aligned} \tag{52}$$

$$\begin{aligned} \mu((-\infty,-\gamma]) &= \mathbb{E}\mathbf{1}_{F_S\leq\gamma} \leq \mathbb{E}\left[\mathbf{1}_{F_S\leq 0}\mathbf{1}_{\varphi_-(L_S)\geq\gamma}\right] \leq \mathbb{E}\left[\mathbf{1}_{F_S\leq 0}\mathbf{1}_{\varphi_-(L_T)\geq\gamma}\right] \\ &= \mu((-\infty,-\gamma]) + \mathbb{E}\left[\left(\mathbf{1}_{F_S\leq 0}-\mathbf{1}_{F_T\leq 0}\right)\mathbf{1}_{\varphi_-(L_T)\geq\gamma}\right], \end{aligned} \tag{53}$$

from which it follows that (cf. (52)) $\xi_\lambda \geq 0$ and likewise $\eta_\gamma := \mathbb{E}[(\mathbf{1}_{F_S\leq 0}-\mathbf{1}_{F_T\leq 0})\mathbf{1}_{\varphi_-(L_T)\geq\gamma}] \geq 0$. Recall that μ has no atoms and therefore the functions $\varphi_{+/-}$ are strictly increasing.[1] It follows that $\xi_\lambda = -\eta_{\varphi_-(\psi_+(\lambda))}$ and thus $\xi_\lambda = \eta_\gamma = 0$ for all $\lambda, \gamma > 0$. This in turn signifies that we had equalities instead of inequalities everywhere in (52) and (53) and thus $L_T = L_S$, $F_S = \varphi_+(L_T)$ on $F_S > 0$, and $F_S = -\varphi_-(L_T)$ on $F_S < 0$. In consequence $T = S$. This ends the proof of Theorem 1.

7.2. *Proof of the embedding for positive functionals*

We now prove Theorem 4. We start with its first part and consider μ with no atom in zero. We note that ψ_μ is increasing and $\psi_\mu(y) < \infty$ if $y \in \operatorname{supp}(\mu)$. Indeed, we have then $\overline{\mu}(y) > 0$ and $n_F([y,\infty)) > 0$ so that $\psi_\mu(y) \leq n_F([y,\infty))^{-1}(\frac{1-\overline{\mu}(y)}{\overline{\mu}} - \log(\overline{\mu}(y)))$. Secondly, note that $\psi_\mu(\infty) = \infty$. Indeed, for $0 < \kappa < y < b_\mu$, we have $\psi_\mu(y+) \geq n_F([\kappa,\infty))^{-1}\log\left(\frac{\overline{\mu}(\kappa)}{\overline{\mu}(y+)}\right) \to \infty$ as $y \to b_\mu$.

We will use our previous study from Section 7.1. Recall the notation from Section 3. If we put $\psi_- = 0$, so that $\varphi_- = \infty$ then $T^F_{\varphi_\mu} = T^F_{\varphi_-,\varphi_+}$, displayed in (9), with $\varphi_+ = \varphi_\mu$. For simplicity we write T for $T^F_{\varphi_\mu}$. The calculation of the law of L_T given in (33) is valid and with a change of variables it is seen that $L_T < \infty$ a.s. and thus the stopping time T is a.s. finite for any probability measure μ with $\mu(\{0\}) = 0$. This follows also, as $\psi_\mu(\infty) = \infty$, from $\overline{\mu}(y) = \mathbb{P}(L_T \geq \psi_\mu(y))$ proved below.

Let ν be the non-atomic part of μ, $\mathrm{d}\nu(x) = \mathbf{1}_{\mu(\{x\})=0}\mathrm{d}\mu(x)$ and $0 < y_1 < y_2 < \ldots$ the atoms of μ. We have to verify that for all $u \geq 0$, $\overline{\mu}(u) = \mathbb{P}(F_T \geq u)$. We have $\mathbb{P}(F_T \geq u) = \mathbb{P}(\varphi_\mu(L_T) \geq u) = \mathbb{P}(L_T \geq \psi_\mu(u))$ as ψ_μ and φ_μ are taken respectively left- and right- continuous. It suffices thus to show that $\overline{\mu}$ and $\mathbb{P}(L_T \geq \psi_\mu(u))$ have the same jumps and that they satisfy the same differential equation for all u such that $\mu(\{u\}) = 0$ and $u \in \operatorname{supp}(\mu)$ (since the measures $\mathrm{d}\psi_\mu$ and $\mathrm{d}\mu$ have the same support). Observe that u with $u \in \operatorname{supp}(\mu)$, $\mu(\{u\}) = 0$ are precisely of the form $u = \varphi_\mu(y)$ with $\psi_\mu(\varphi_\mu(y)) = y$.

[1] This is not necessary for minimality of T but simplifies the proof.

Let $u = \varphi_\mu(y)$ such that $\mu(\{u\}) = 0$ so that $\psi_\mu(u) = y$. Note that $\mathrm{d}\nu(u) = \mathrm{d}\mu(u)$ and $\mathrm{d}y = \mathrm{d}(\psi_\mu(u)) = \mathrm{d}\mu(u)[\overline{\mu}(u)n_F([u,\infty))]^{-1}$. We have thus

$$\begin{aligned}\mathrm{d}\left(\mathbb{P}(L_T \geq \psi_\mu(u))\right) &= \mathrm{d}\mathbb{P}(L_T \geq y) = -\mathbb{P}(L_T \geq y)n_F([\varphi_\mu(y),\infty))\mathrm{d}y \\ &= \mathbb{P}(L_T \geq \psi_\mu(u))n_F([u,\infty))\mathrm{d}\psi_\mu(u) \\ \mathrm{d}\left(\overline{\mu}(u)\right) &= -\overline{\mu}(u)n_F([u,\infty))\mathrm{d}y = -\overline{\mu}(u)n_F([u,\infty))\mathrm{d}\psi_\mu(u),\end{aligned} \tag{54}$$

which shows that the two functions $\overline{\mu}(\cdot)$ and $\mathbb{P}(L_T \geq \psi_\mu(\cdot))$ satisfy the same differential equation on the required set. It remains to show that $\mu(\{y_i\}) = \mathbb{P}(\psi_\mu(y_i) \leq L_T < \psi_\mu(y_i+)), i \geq 1$. Suppose we know this already for $i \leq j-1$ for some $j \geq 2$. Combined with the discussion above, this yields $\overline{\mu}(y_j) = \mathbb{P}(L_T \geq \psi_\mu(y_j))$. Note also that for $y \in [\psi_\mu(y_j), \psi_\mu(y_j+)), \varphi_\mu(y) = y_j$ and let $\Delta\psi_\mu(y_j) = \psi_\mu(y_j+) - \psi_\mu(y_j) = \ln\left(\frac{\overline{\mu}(y_i)}{\overline{\mu}(y_i+)}\right)n_F([y_i,\infty))^{-1}$. By (33) we have

$$\begin{aligned}\mathbb{P}(\psi_\mu(y_j) \leq L_T < \psi_\mu(y_j+)) &= \mathbb{P}(L_T \geq \psi_\mu(y_j)) - \mathbb{P}(L_T \geq \psi_\mu(y_j+)) \\ &= \mathbb{P}(L_T \geq \psi_\mu(y_j))\left(1 - \exp\{-\Delta\psi_\mu(y_j)n_F([y_j,\infty))\}\right) \\ &= \overline{\mu}(y_j)\left(1 - \frac{\overline{\mu}(y_j+)}{\overline{\mu}(y_j)}\right) = \overline{\mu}(y_j) - \overline{\mu}(y_j+) \\ &= \mu(\{y_j\}),\end{aligned} \tag{55}$$

which ends the proof of the embedding. It remains to verify that T is minimal. From the definition of T in (18), as φ_μ is non-decreasing, it is clear that $\sup_{t \leq T^F_{\varphi_\mu}} F_t = F_{T^F_{\varphi_\mu}}$. Let S be a stopping time such that $F_S \sim F_T$ and $S \leq T$. We have $\sup_{t\leq S} F_t \leq \sup_{t\leq T^F_{\varphi_\mu}} F_t = F_{T^F_{\varphi_\mu}}$ and thus $0 \leq F_S \leq F_T$ which, together with $F_T \sim F_S$, implies that $F_S = F_T$ a.s. and thus $S = T$ a.s. This ends the proof of the first part of Theorem 4.

To finish the proof, we now argue the embedding for a probability measure μ with an atom in zero. It is an easy consequence of what we have obtained so far. Recall that we defined a new probability measure on $(0,\infty]$ transferring the atom in zero to infinity, $\tilde{\mu} = \mu - \varsigma(\delta_0 - \delta_\infty)$, where $\varsigma = \mu(\{0\})$. We thus have $\varsigma = \overline{\tilde{\mu}}(\infty) > 0$ which implies $\psi_{\tilde{\mu}}(\infty) = \delta < \infty$. In turn, $\varphi_{\tilde{\mu}}(x) = \infty$ for $x \geq \delta$ and $\mathbb{P}(L_{T^F_{\varphi_{\tilde{\mu}}}} = \infty) = \mathbb{P}(L_{T^F_{\varphi_{\tilde{\mu}}}} \geq \delta) = \varsigma$, and $F_{T^F_{\varphi_{\tilde{\mu}}}}\mathbf{1}_{L_{T^F_{\varphi_{\tilde{\mu}}}}<\delta} \sim \tilde{\mu}|_{(0,\infty)} \sim \mu|_{(0,\infty)}$. It suffices now to observe that for $R = \inf\{t : L_t = \delta\}$ we have $F_R = 0$ as the support of $\mathrm{d}L_t$ is contained in the set of zeros of (F_t), to conclude that $F_{R\wedge T^F_{\varphi_{\tilde{\mu}}}} \sim \mu$.

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